

Proton rms-radii from low- q power expansions?

Ingo Sick and Dirk Trautmann

Dept. für Physik, Universität Basel, CH4056 Basel, Switzerland*

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Several recent publications claim that the proton charge *rms*-radius resulting from the analysis of electron scattering data restricted to *low* momentum transfer agrees with the radius determined from muonic hydrogen, in contrast to the radius resulting from analyses of the full (e,e) data set which is $0.04 fm$ larger. Here we show why these publications erroneously arrive at the low radii.

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Introduction. The determination of the *rms*-radius R of the proton charge distribution has recently attracted much attention. While standard analyses of electron-proton scattering data yield $0.879 \pm 0.009 fm$ [1], the Lamb shift measurement in muonic hydrogen gave $0.8409 \pm 0.0004 fm$ [2]; this represents a $\approx 5\sigma$ discrepancy. The radii from electron scattering near $0.88 fm$ come from analyses that fit with excellent χ^2 the *world* cross section and polarization transfer data up to large momentum transfer q , $5 fm^{-1}$ to $12 fm^{-1}$ [3–8]. Recently, 3 publications [9–11] which restrict the analysis to the *low- q* data, with $q_{max} = 0.7, 0.9$ and $1.6 fm^{-1}$ respectively, find R in the $0.84 fm$ neighborhood, *i.e.* compatible with the radius from muonic hydrogen. In this paper, we show why these analyses, which yield values of $R \approx 0.04 fm$ lower than refs. [3–8], have led to erroneously low values.

Power series expansion. In terms of the electric Sachs form factor $G_e(q)$ the proton charge *rms*-radius R is defined via the slope of $G_e(q^2)$ at $q^2 = 0$. It therefore seems natural to parameterize $G(q)$ in a power series

$$G_e(q) = 1 + q^2 a_2 + q^4 a_4 + q^6 a_6 + \dots \quad (1)$$

where $R^2 = -6a_2$. Non-relativistically, $a_4 = \langle r^4 \rangle / 120$ and $a_6 = -\langle r^6 \rangle / 5040$ are given by the higher moments of the charge density distribution. The rationale behind an analysis restricted to data with *low* maximum momentum transfer q_{max} : at low enough q the terms proportional to q^{2n} with $n > 1$ (or in some cases $n > 2$) can be neglected, so a linear (quadratic) fit of the data in terms of powers of q^2 should suffice. Low order (one parameter) fits in terms of derived functions as *e.g.* a dipole, $G(q) = 1/(1+q^2 b_2)^2$, follow the same rationale, although these parameterizations do implicitly contain higher $q^{2n} a_{2n}$ contributions as fixed by the analytical shape of the parameterization.

Problems with expansions of the proton form factors in terms of q^{2n} have been recognized earlier [12]. Due to the peculiar shape of the proton form factor — approximately a dipole — and the peculiar shape of the

corresponding charge density — approximately an exponential — the moments $\langle r^{2n} \rangle$ for $n \geq 2$ grow unusually fast with increasing order n . In the form factor $G(q)$ the moments $\langle r^{2n} \rangle$ are tightly coupled and give contributions of alternating signs. In an expansion with small n ($n = 1, 2$) the values found for $\langle r^{2n} \rangle$ depend on the maximum n and the value of the maximum momentum transfer q_{max} employed, and always yield too small $\langle r^2 \rangle$. This has recently been shown by Kraus *et al.* [13] who quantitatively demonstrate the pitfalls of fits with low order power series by analyzing pseudo-data generated with known R . They show that *e.g.* a linear fit in q^2 with $q_{max} = 0.7 fm^{-1}$ as employed in [9, 10] produces a value of R which is low by $0.04 fm$.

This result of Kraus *et al.* can qualitatively be understood. When terminating the series eq.(1) with the q^2 -term, one implicitly posits $\langle r^4 \rangle = 0$. As $\langle r^2 \rangle \approx 0.7 fm^2$ this implies a charge density that is positive at small r (charge proton $+e$), but has a negative tail at large r ; due to the larger weight in the r^4 -term the tail can reduce $\langle r^4 \rangle$ to 0. This negative tail of course also affects $\langle r^2 \rangle$, and leads to the systematically low values of R . The same happens *mutatis mutandis* with truncations at higher order [13].

The second, obvious, problem with very low q : the finite size effect (FSE) $1 - G_e(q)$ decreases like q_{max}^2 . Already at the $q \approx 0.8 fm^{-1}$ of maximal sensitivity of the data to R (see below) the FSE $\approx q^2 R^2 / 6$ amounts to 0.09 only. The smallness of the FSE emphasizes that fits used to extract R must reach the minimal χ_{min}^2 achievable, a visually good fit is not enough: a change of R of 1% corresponds to a systematic change of G_e of only 0.0015 (0.17% of G_e), a difference that is far below the resolution of typical plots of $G_e(q)$ [9–11].

The sensitivity of the data to R is shown in Fig.1 which results from a notch test employing SOG fits of the *world* data (for recent reference to notch tests see [14]). When exploiting only part of the range of $q \leq 1.5 fm^{-1}$, one loses part of the experimental information on R ; analyses which limit the data to *e.g.* $0.8 fm^{-1}$ as done in refs.[9, 10] then ignore half of the data sensitive to R . Restriction to a subset of the *world* data only amplifies this problem.

* Ingo.Sick@unibas.ch, Dirk.Trautmann@unibas.ch

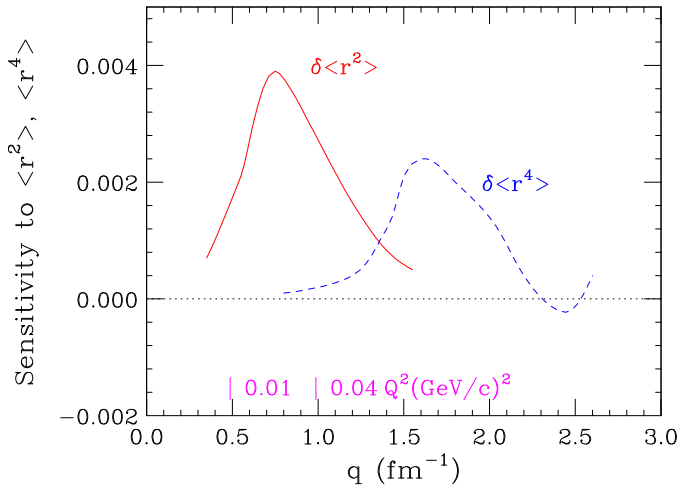


FIG. 1. (Color online) Sensitivity (arbitrary units) to the moments $\langle r^2 \rangle$ and $\langle r^4 \rangle$ obtained from fits of the *world* data.

Contribution of higher moments. For a more detailed discussion of the problems with eq.(1), we start from the values of a_2, a_4, \dots determined by Bernauer *et al.* [15] via a power-series fit (with a χ^2 as low as a spline fit) to the Mainz data for $q_{max} = 5 \text{ fm}^{-1}$. One might hope that, due to the large q_{max} and the high order $2n = 20$ employed, the values of the lowest moments of interest here should not be affected seriously by the above-mentioned problems [12]. Fig.2 shows the percent contribution of the a_4 to a_{10} terms to the FSE. Also indicated is the uncertainty in the FSE due to a (very optimistic) uncertainty of 0.2% in the experimental $G_e(q)$.

This figure shows several features:

1. At the q 's used in the 'low- q fits' referred to above, with $q_{max} = 0.72 - 0.9 \text{ fm}^{-1}$, the contribution of the q^4 -term to the FSE $\approx q^2 R^2 / 6$ amounts to 10–15% at the upper limit of the q -range where FSE is most sensitive to R . This shows immediately and without further calculation that neglecting this contribution in a linear fit in terms of q^2 must yield a value of R^2 which is low by a comparable percentage.

2. Even the contribution of the q^6 -term is not entirely negligible (15% of the q^4 -term at $q = 0.9 \text{ fm}^{-1}$); when attempting to determine a_4 from a fit quadratic in q^2 a wrong value results if the contribution of the q^6 -term is not accounted for.

3. Restriction of q_{max} to extremely low values, such as to justifiably neglect the q^4 -term and maintain an accuracy of 1% in R , would require $q_{max} < 0.35 \text{ fm}^{-1}$. At these values of q , the FSE is < 0.015 , and the typical error bars of $G_e(q)$ would yield huge uncertainties in the FSE contribution, hence R^2 (see dashed curve).

Fig.2 makes it obvious that the low- q fits of refs.[9, 10], which neglect the q^4 -contribution, must find wrong values for R due to the omitted q^4 term (for a quantitative analysis see below). Fig.2 also shows, without further calculation, that for $q \leq 1.6 \text{ fm}^{-1}$ the information content

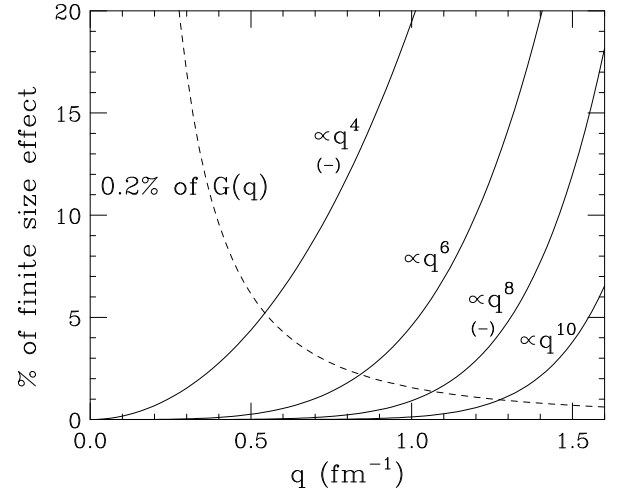


FIG. 2. The solid curves show the relative contribution (in %) of the q^{2n} terms to the finite size effect FSE in $G_e(q)$. The dashed curve shows the relative contribution of an 0.2% uncertainty of the experimental $G_e(q)$. For comparison: the q_{max} of the fits linear in q^2 (dipole) of refs. [9, 10]([11]) amount to 0.72, 0.90 and 1.6 fm^{-1} , respectively.

of the data is 4–5 parameters (moments) which hardly can be represented correctly by a one-parameter form-factor such as employed by Horbatsch+Hessels[11] (for a quantitative discussion see below).

Higher moments from world data. As was pointed out in [12] and quantitatively demonstrated in [13] the determination of the lowest moments via a power-series fit is not very reliable and for the higher n dependent on the cut-off in n . We therefore have made an independent determination.

We use the *world* data up to the maximum momentum transfer available for G_e , 10 fm^{-1} (not including the data of ref.[15] which show systematic differences [3]). This data set, which comprises 603 cross sections and polarization transfer points, is corrected for 2-photon exchange effects [16] and fitted with a Fourier transform of Laguerre functions of order 11 for both $G_e(q)$ and $G_m(q)$. Laguerre functions¹ are particularly well suited as

- They provide an orthonormal basis which makes multi-parameter fits very efficient (even if the polynomials are not strictly orthogonal over the limited q -range of the data).
- They have a controlled behavior at large radii r due to the $e^{-\gamma r}$ weight function, a consideration which is particularly important [20] when addressing higher moments (an aspect shared with the parameterizations of the Vector Dominance Model VDM).
- They provide values for the moments insensitive to the

¹ For similar expansions see [17–19]

cutoff in the number of terms employed; the moments $\langle r^{2n} \rangle$ are given by the lowest $2n + 3$ coefficients.

The set of data can be reproduced with a χ^2 of 542 with 548 degrees of freedom when the normalizations of the individual data sets are floated. When keeping the normalizations at their measured values, and *without* increasing the error bars due to systematic error of the normalizations, the χ^2 amounts to 783 with 580 degrees of freedom. These χ^2 values are excellent given a set of data measured over some 50 years. The resulting values for $\langle r^4 \rangle$ are 2.01 ± 0.05 (1.99) fm^4 . The quality of the fit and the values of the moments are very close to the ones obtained using SOG [21] ($\langle r^4 \rangle = 2.03$) or a VDM-type parameterization ($\langle r^4 \rangle = 2.01$). We have verified that a variation of q_{max} between 7 and $12 fm^{-1}$ and a variation of n between 10 and 13 changes $\langle r^4 \rangle$ by $< 0.03 fm^4$. Distler *et al.* [22] obtained $2.59 \pm 0.19 \pm 0.04$ from a mix of two form factor parametrizations fit separately to low- q [15] and high- q [23] data. With these preliminaries we are in the position to quantitatively discuss the recent low- q fits.

Fits to very-low q data. Higinbotham *et al.* [10] perform a linear fit in q^2 to a subset of the data available, the form factors of Mainz80+Saskatoon74 [24, 25]. For their highest q_{max} of $0.9 fm^{-1}$, which yields the result with the smallest uncertainty, they find² $R = 0.844 \pm 0.014 fm$. From this the authors conclude that R agrees with the value of $0.84 fm$ from muonic hydrogen. When repeating exactly the same analysis, but adding in the q^4 and q^6 contributions using the higher moments from the fit to the high- q data, one finds a reduced χ^2 (*i.e.* χ^2 per degree of freedom) which is 11% smaller and a radius R of $0.899 fm$. This R disagrees with the muonic value, and agrees with the above-cited R 's in the $0.88 fm$ region.

Higinbotham *et al.* also perform a fit quadratic in q^2 , and find a radius of $0.873 \pm 0.039 fm$. This agrees with the radii in the $0.88 fm$ region, although, as the authors want to see it, the value is “within one σ of the muonic result”. The uncertainty of $\pm 0.039 fm$ illustrates the large error bars resulting from the restriction of the analysis to a fraction of the q -region sensitive to R (see Fig.1) and the large uncertainty of $\langle r^4 \rangle$ due to the truncation in q . When using, instead of the $\langle r^4 \rangle = 1.32 \pm 0.96$ of Higinbotham *et al.*, the value 2.01 ± 0.05 we know from the fit to the high- q data, the result for R becomes $0.901 fm$, with a smaller error bar of $0.010 fm$.

Griffioen *et al.* [9] analyze part of the cross sections of [7] for $q < 0.72 fm^{-1}$ using eq.(1) including terms up to a_4 . They use a low- q parameterization for G_m/G_e and take the shortcut of ignoring the free relative normalizations of the individual data sets³. They find an

rms-radius of $0.850 \pm 0.019 fm$ and conclude that this value is consistent with the muonic hydrogen result of $0.84 fm$. Repeating their fit, but using the a_4 determined much better from the high- q fit, yields a radius of $0.877 \pm 0.008 fm$, with lower χ^2 and a significantly smaller error bar. This result agrees with the $0.88 fm$ -type results, and disagrees with the radius from muonic hydrogen.

Griffioen *et al.* also perform fits up to order q^6 , with a_4, a_6 -values as given by simple models for the proton charge density (uniform, exponential, gaussian) which all produce the same χ^2 ; the resulting R -values are linearly correlated with a_4 . Extrapolating these values linearly to the value of a_4 given by the fit to high- q data yields $R = 0.876 \pm 0.008 fm$, again in agreement with the R 's in the $0.88 fm$ region.

The bottom line: all the low- q fits of refs.[9, 10] yield radii in the $0.88 fm$ region once the higher moments of the charge density — which *are* non-zero but ignored (or poorly fixed in the low- q fits due to the truncation of the series in n of q_{max}) — are properly accounted for.

Fits to not-so-low q data. Horbatsch and Hessels [11] employ the cross sections of ref.[7] up to a q_{max} of $1.6 fm^{-1}$. They parameterize the form factors via a 1-parameter dipole expression for both G_e and G_m . Their fit yields a reduced χ^2 of 1.11, and a (charge) *rms*-radius $R = 0.842 \pm 0.002 fm$. From this, together with other fits which yield radii near $0.89 fm$, the authors conclude that R is in the range $0.84 - 0.89 fm$, *i.e.* could be compatible with the radius from muonic hydrogen.

Fig.2 shows that for $q_{max} = 1.6 fm^{-1}$ the moments up to at least $2n = 10$ are important to get the full FSE. It is highly unlikely that the one-parameter dipole contains the mix of q^{2n} -terms for $2n = 4...10$ appropriate for the proton. Indeed, expansion of the dipole in terms of powers of q^2 shows that the numerically largest difference to the power-series fit of [15] results from the contribution of the $\langle r^4 \rangle$ term. This difference in $\langle r^4 \rangle$ alone would lead, at the $q = 0.85 fm^{-1}$ of maximal sensitivity to R , to a difference ΔG_e of 0.0081 corresponding to 9.5% in the FSE, hence R^2 (causing the systematic deviations just visible in Fig.3 of [11]). The same consideration applies to the parameterization of $G(q)$ as a (one-parameter) linear function $1 - cz$ with $z = (\sqrt{t_c - t} - \sqrt{t_c})/(\sqrt{t_c - t} + \sqrt{t_c})$ and $t = -q^2$. The lacking flexibility of the fit function, causing systematic differences between data and fit and a χ^2 larger than the one of already published fits, also affects the results from the high- q fits of [9, 10].

For the fits of Horbatsch and Hessels it is not practical to correct for the effect upon R of the incorrect higher q^{2n} -terms as we did above for the analyses of refs.[9, 10]; too many terms $2n = 4...10$ would contribute. In order to demonstrate the importance of their effect we rather quote the result of a Laguerre-function fit (4 terms each for G_e and G_m) to exactly the same data, yielding a lower reduced χ^2 of 1.045 and a (charge) *rms*-radius $R = 0.884 \pm 0.016 fm$. Due to the lacking flexibility the parameterization of Horbatsch+Hessels has a χ^2 that is

² Including Coulomb distortion would have increased R by $\approx 0.01 fm$ [26]

³ Correct treatment of the normalizations of the data sets of [7], which are individually floating, would have increased the uncertainty of R by a factor 1.6.

higher by 50! From such a “fit”, that is some 7 σ ’s away from a genuine best-fit, one obviously cannot get a significant value for R .

Conclusion. The moments $\langle r^{2n} \rangle$ of the proton for

$n > 1$ are there, and they are *known* to be large. Ignoring their strong correlation with R [9–11] leads to wrong results for the proton *rms*-radius.

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